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An Application of the Duality Theorem of Linear Programming to Testing Hypotheses

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Summary: The problem of testing a composite hypthesis versus a composite alternative is considered from the Neyman-Pearson viewpoint where the parameter space and sample space are both finite. The problem is shown to be a linear programming problem and the duality theorem is invoked to obtain results concerning least favorable a priori weighting functions and their use in obtaining optimal tests.

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1. <u>Introduction</u>: In this paper we discuss some specialized aspects of the Neyman-Pearson problem in testing hypotheses. In particular, we are concerned with the method of weight functions which are introduced to reduce a composite hypothesis to a simple hypothesis, a composite alternative to a simple alternative. It has been pointed out by Wald [9], pages 199 to 207 Lehmann [7], pages 2-9 to 2-15, and others that if one has a testing hypotheses problem and one collapses it to a simple hypothesis versus a single alternative in such a manner that an α_0 -most powerful test for the auxiliary problem has a power curve that satisfies certain properties, then this test is α_0-optimal for the original problem. That this collapsing can be achieved follows in special cases from the interrelationships in decision theory involving Bayes solutions and the admissible class. These results in decision theory, in turn, stem from some basic theorems on the theory of convex bodies or simple consequences of these theorems involving the theory of games. Furthermore, the theory of games is intimately related to the theory of linear programming (eg., the fundamental theorem in both cases is a direct consequence of the theorem that non-intersecting convex sets can be separated by a hyperplane). We wish to point out that for some simple problems of the Neyman-Pearson testing hypotheses variety the theory of linear programming seems ideally suited and that from a single technique many important results immediately follow.

2. Summary of the duality theorem of linear programming:

For completeness sake, we first summarize some necessary results from the theory of linear programming. For self-contained references to proofs of the results of this section see the papers **Dantzig [1], Gale [3], Gerstenhaber [5], and Gale, Kuhn, Tucker [4].** As datum of the problem let us be given an array of reals numbers $\|a_{ij}\|$, $i = 1,2,\ldots m$; $j = 1,2,\ldots,n$; and real numbers $b_1, b_2, \ldots, b_n$ and $c_1, c_2, \ldots, c_m$.

Problem 2.1: Let

$$F = \{\underline{x} = (x_1, x_2, \ldots, x_n): \quad x_j \geqslant 0 \text{ for } j = 1,2,\ldots,n \text{ and}$$

$$\sum_j a_{ij}\, x_j \leqslant c_i \text{ for } i = 1,2,\ldots,m\} \;;$$

to find $\underline{x}^{(0)} \in F$ such that $\underline{x}^{(0)}$ is a maximizer of $\sum_j b_j x_j$ (i.e., such that $\sum_j b_j x_j \leqslant \sum_j b_j x_j^{(0)}$ all $\underline{x} \in F$).

Problem 2.2: Let

$$H = \{\underline{y} = (y_1, y_2, \ldots, y_m): \quad y_i \geqslant 0 \text{ for } i = 1,2,\ldots,m \text{ and}$$

$$\sum_i y_i\, a_{ij} \geqslant b_j \quad \text{for } j = 1,2,\ldots,n\} \;;$$

to find $\underline{y}^{(0)} \in H$ such that $\underline{y}^{(0)}$ is a minimizer of $\sum_i c_i y_i$ (i.e., such that $\sum_i c_i y_i \geqslant \sum_i c_i y_i^{(0)}$ all $\underline{y} \in H$).

Theorem 1:

a) There exists a solution to Problem 2.1 if and only if there exists a solution of Problem 2.2.

b) If $\underline{x} \in F$ and $\underline{y} \in H$ then $\sum_j b_j x_j \leq \sum_i c_i y_i$

c) If $\underline{x}^{(0)}$ and $\underline{y}^{(0)}$ are solutions of Problems 2.1 and 2.2 respectively, then $\sum_j b_j x_j^{(0)} = \sum_i c_i y_i^{(0)}$.

d) If $\underline{x}^{(1)} \in F$, $\underline{y}^{(1)} \in H$ and $\sum_j b_j x_j^{(1)} \geqslant \sum_i c_i y_i^{(1)}$ then $\underline{x}^{(1)}$ and $\underline{y}^{(1)}$ are solutions to Problem 2.1 and 2.2 respectively.

Problems 2.1 and 2.2 are said to be dual to each other. Note that these problems are simultaneously solved if we can satisfy the hypothesis of part d) of the theorem which does not require a minimization or maximization feature. This combined version is called the symmetric form of the problem.

3. Simple hypothesis vs. single alternative:

We apply the results of section 2 to prove a simple case of the well-known Neyman-Pearson Lemma. Our interest is not in the result of the theorem but in the techniques of the proof which will be utilized in sections 4 and 5.

Let ω_0 and ω_1 be two states of nature; let S be the sample space with a finite set of points $x_1, x_2, \ldots, x_n$ and let $P\{x_i \mid \omega_0\} = f_i$ and $P\{x_i \mid \omega_1\} = g_i$ for $i = 1,2,\ldots,n$.

Let $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ be the randomized decision rule which asserts that if x_i is observed we should accept ω_0 and ω_1 with probabilities $1 - \phi_i$ and ϕ_i respectively (naturally $0 \leq \phi_i \leq 1$ for $i = 1,2,\ldots,n$). Each decision rule ϕ is appraised by its error vector $(\alpha(\phi), \beta(\phi))$ where $\alpha(\phi) = \sum_i f_i \phi_i$ (the probability of the error of type I) and $\beta(\phi) = 1 - \sum_i g_i \phi_i$ (the probability of the error of type II). The Neyman-Pearson problem is to find $\phi^{(0)}$ such that for a preassigned α_0, (the significance level) $\phi^{(0)}$ is a minimizer of $\beta(\phi)$ among the class of ϕ such that $\alpha(\phi) \leq \alpha_0$. We restate this problem as follows:

Problem 3.1: Let F be the set of all $\phi = (\phi_1, \ldots, \phi_n)$ such that

$$
\begin{array}{lll}
 & \text{i)} & \phi_i \geq 0, \quad i = 1,2,\ldots,n \\
(3.1) & \text{ii)} & \phi_i \leq 1, \quad i = 1,2,\ldots,n \\
 & \text{iii)} & \alpha(\phi) = \sum_i f_i \phi_i \leq \alpha_0, \quad (0<\alpha_0<1);
\end{array}
$$

to find $\phi^{(0)} \in F$ which is a maximizer of

$$(3.2) \qquad 1 - \beta(\phi) = \sum_i g_i \phi_i .$$

The dual problem is then

Problem 3.2: Let H be the set of all n+1-tuples $(\xi_1, \xi_2, \ldots, \xi_n, \rho)$ such that

$$(3.3) \qquad \begin{array}{ll} \text{i)} & \xi_i \geq 0, \quad i = 1,2,\ldots,n \quad \text{and} \quad \rho \geq 0, \\ \text{ii)} & \xi_i + \rho f_i \geq g_i, \quad i = 1,2,\ldots,n; \end{array}$$

to find $(\xi_1^{(0)}, \ldots, \xi_n^{(0)}, \rho^{(0)}) \in H$ which is a minimizer of

$$(3.4) \qquad \sum_i \xi_i + \alpha_0 \rho .$$

Without any loss of generality we can label the sample points $x_1, x_2, \ldots, x_n$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ where $\lambda_i = g_i / f_i$ (if $g_i = f_i = 0$ we merely exclude x_i from the sample space), $i = 1,2,\ldots,n$; furthermore, we can assume that $f_i > 0$ for $i = 2,3,\ldots,n$.

<u>Theorem 2</u>: Let ϵ be a real number where $0 \leqslant \epsilon \leqslant 1$ and let u be a non-negative interger such that

(3.5) $$f_1 + f_2 + \dots + f_{u-1} + \epsilon f_u = \alpha_0 .$$

a) If $(\xi_1^{(0)}, \dots, \xi_n^{(0)}, \rho^{(0)})$ is defined as follows:

i) $\rho^{(0)} = \lambda_u$ $(\rho^{(0)} \neq \infty$ since $\alpha_0 > 0)$

(3.6) ii) $\xi_i^{(0)} + \rho^{(0)} f_i = g_i, \quad i \leq u-1$

iii) $\xi_i^{(0)} = 0$ for $i \geqslant u$,

then it is a solution of Problem 3.2.

b) If $\emptyset^{(0)}$ is such that

$$\begin{aligned} \emptyset_i^{(0)} &= 1 \quad \text{for } i < u \\ &= \epsilon \quad \text{for } i = u \\ &= 0 \quad \text{for } i > u \end{aligned}$$

then it is a solution of Problem 3.1.

<u>Proof</u>: To see that $(\xi_1^{(0)}, \dots, \xi_n^{(0)}, \rho^{(0)}) \in H$ we note the following: $\lambda_u \geqslant 0$; since $\lambda_u \leq \lambda_i$ for $i \leqslant u-1$ it follows that $\lambda_u f_i \leq g_i$ and $\xi_i^{(0)} \geqslant 0$ for all i; since $\lambda_u \geqslant \lambda_i$ for $i \geqslant u$ it follows that $\lambda_u f_i \geqslant g_i$ for $i \geqslant u$ and therefore $\xi_i^{(0)} + \rho^{(0)} f_i \geqslant g_i$, all i. By

the definition of the pair (ϵ, u) it also follows that $\emptyset^{(0)} \in F$. Now from 3.6 ii) and 3.5

$$\Sigma_i \xi_i^{(0)} + \alpha_0 \rho^{(0)} = g_1 + g_2 + \cdots + g_{u-1} + \epsilon g_u$$

$$= \Sigma_i g_i \emptyset_i^{(0)} = 1 - \beta(\emptyset^{(0)}),$$

and from Theorem 1, part d), Theorem 2 follows.

A geometrical interpretation for Problem 3.2 is rather interesting. We illustrate it in Figure 1 below for n = 2. The generalization to arbitrary (finite) n is obvious. Let us consider the points (f_1, f_2), (g_1, g_2) and the displaced positive orthant with vertex at (g_1, g_2). We assert that Problem 3.2 is equivalent to the

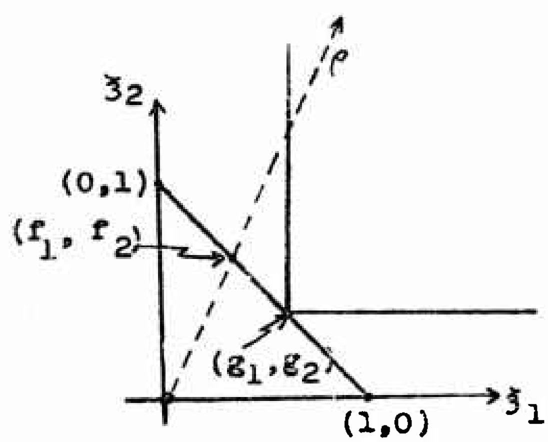

Figure 1

problem of finding the minimum distance from the origin to the displaced positive orthant where the distance is taken under the

constraint that one can only move parallel to the coordinate axes or along a ray with direction numbers (f_1, f_2); furthermore the unit of measurement along this ray is such that the distance from (0,0) to (f_1, f_2) is α_0. Thus for a given configuration the minimum path depends on the α_0 (the smaller the α_0 the more $\rho\alpha_0$ units one should travel along the ray). If $\alpha_0 < f_i$ then one should not move along ξ_i, $(i = 1,2)$ but rather along the ray.

4. The composite hypothesis versus the single alternative.

Now suppose that ω_0 is a composite hypothesis with atomic states $\omega_{01}, \omega_{02}, \ldots, \omega_{0r}$. Let $f_{ij} = P\{x_i \mid \omega_{0j}\}$ and again let $g_i = P\{x_i \mid \omega_1\}$. Let $\underline{\phi} = (\phi_1, \phi_2, \ldots, \phi_n)$ again be the randomized decision which accepts ω_0 and ω_1 with probabilities $1 - \phi_i$ and ϕ_i, respectively, if x_i is observed.

Problem 4.1: Let F be the set of all $\underline{\phi}$ such that

$$\text{i)} \quad \phi_i \geqslant 0, \quad i = 1,2,\ldots,n,$$

$$(4.1) \qquad \text{ii)} \quad \phi_i \leqslant 1, \quad i = 1,2,\ldots,n,$$

$$\text{iii)} \quad \alpha_j(\underline{\phi}) \equiv \sum_i f_{ij}\,\phi_i \leqslant \alpha_0 \quad \text{for} \quad j = 1,2,\ldots,r;$$

to find $\underline{\phi}^{(0)} \in F$ which is a maximizer of

$$(4.2) \qquad 1 - \beta(\underline{\phi}) = \sum_i g_i\,\phi_i .$$

The dual problem is then:

Problem 4.2: Let H be the set of all n+r - tuples $(\xi_1, \ldots, \xi_n, \rho_1, \ldots, \rho_r)$ such that

$$(4.3) \qquad \begin{array}{l} \text{i)} \quad \xi_i \geqslant 0, \quad \rho_j \geqslant 0, \quad i = 1,2,\ldots,n \text{ and } j = 1,2,\ldots,r, \\ \text{ii)} \quad \xi_i + \sum_j \rho_j f_{ij} \geqslant g_i, \quad i = 1,2,\ldots,n; \end{array}$$

to find $(\xi_1^{(0)},\ldots,\xi_n^{(0)},\rho_1^{(0)},\ldots,\rho_r^{(0)}) \in H$ which is a minimizer of

$$\Sigma_i \, \xi_i + \alpha_0 \, \Sigma_j \, \rho_j \, .$$

Let $\rho = \Sigma_j \, \rho_j$ and $\eta = (\rho_1^*,\ldots,\rho_r^*)$ where $\rho_j^* = \rho_j/\rho$ for $j = 1,2,\ldots,r$. There might be some difficulty if $\rho = 0$. We assume $\rho \neq 0$ since the case where $\rho = 0$ is trivial and can be cleaned up at the end. Note that η is a probability vector (i.e., $\rho_j^* \geqslant 0$, $\Sigma_j \, \rho_j^* = 1$). Let $f_i(\eta) = \Sigma_j f_{ij} \rho_j^*$. We can now rewrite Problem 4.2 as follows:

Problem 4.2': Let H' be the set of all elements $(\xi_1,\ldots,\xi_n,\rho,\eta)$ such that

(4.5) i) $\xi_i \geqslant 0$, $i = 1,2,\ldots,n$, $\rho \geqslant 0$, and η is a probability vector with r components.

(4.5) ii) $\xi_i + \rho f_i(\eta) \geqslant g_i$, $i = 1,2,\ldots,n$;

to find $(\xi_1^{(0)},\ldots,\xi_n^{(0)},\rho^{(0)},\eta^{(0)}) \in H'$ which is a minimizer of

(4.6) $$\Sigma_i \, \xi_i + \alpha_0 \rho .$$

It is easily checked that $\Sigma_i \, f_i(\eta) = 1$ and $f_i(\eta) \geqslant 0$, $i = 1,2,\ldots,n$ for all probability vectors η. For an arbitrary

but fixed η we can define the dual of Problem 4.2' which in turn has the following auxiliary statistical problem associated with it: test the simple hypothesis $\omega_0(\eta)$ against ω_1 where

$$(4.7) \qquad P\{x_i \mid \omega_0(\eta)\} = f_i(\eta) \text{ and } P\{x_i \mid \omega_1\} = g_i, \quad i = 1,2,\dots,n.$$

If we let $\phi(\eta)$ be the most powerful size α_0 test for the auxiliary problem (i.e., among those ϕ for which $\alpha(\phi) = \Sigma_i f_i(\eta)\phi_i \leqslant \alpha_0$ let $\phi(\eta)$ be a minimizer of $\beta(\phi) = 1 - \Sigma_i g_i \phi_i$) then the results of Section 3 indicate that for the fixed η the element $(\xi_1, \xi_2, \dots, \xi_n, \rho)$ which satisfies (4.5 i) and ii)) and minimizes (4.6) yields a minimum value $1 - \beta(\phi(\eta)) = 1 - \Sigma_i g_i \phi_i(\eta)$. Consequently, we obtain a solution of Problem 4.2' by choosing η to minimize $1 - \beta[\phi(\eta)]$ or equivalently to maximize $\beta(\phi(\eta))$. Let $\eta^{(0)}$ be a maximizer of $\beta[\phi(\eta)]$ taken over all probability vectors. (The existence of such an $\eta^{(0)}$ follows trivially from a compactness argument invoked on Problem 4.1 and thus by Theorem 1a) a solution exists for Problem 4.2 and thus for Problem 4.2'). The element $\eta^{(0)}$ is then said to be <u>a least favorable a priori weighting over ω_0 relative to the significance level α_0</u>.

Let ϕ be a solution of Problem 4.1. Then from Theorem 1 c) we have

$$\sum_i \phi_i^{(0)} \xi_i = 1 - \beta\left[\phi\,(\eta^{(0)})\right] .$$

Since $\alpha_j(\phi^{(0)}) \leqslant \alpha_0$ for $j = 1,2,\ldots,r$, $\phi^{(0)}$ is a most powerful size α_0 test of the auxiliary problem $\omega_0(\eta^{(0)})$ against ω_1. We thus conclude that $\phi^{(0)}$ <u>must be included amongst the class of most powerful size α_0 - tests of $\omega_0(\eta^{(0)})$ against</u> ω_1.

We summarize by means of the following theorem.

<u>Theorem 3</u>: If $\phi^{(0)}$ is a solution of Problem 4.1 then $\phi^{(0)}$ is a most powerful α_0- test for $\omega_0(\eta^{(0)})$ against ω_1 (cf. 4.7) where $\eta^{(0)}$ is a least favorable a priori weighting over ω_0 relative to significance level α_0.

<u>Corollary</u>: For a given a priori weighting, say $\eta^{(1)}$, let $\phi(\eta^{(1)})$ be a most powerful α_0 - test for $\omega_0(\eta^{(1)})$ against ω_1. If $\phi(\eta^{(1)}) \in F$ (cf. Problem 4.1) then $\phi(\eta^{(1)})$ is a solution of Problem 4.1 and $\eta^{(1)}$ is a least favorable a priori weighting over ω_0 relative to significance level α_0.

<u>Proof of Corollary</u>: The proof is an immediate consequence of Theorem 1 (part d) as applied to Problems 4.1 and 4.2'.

The fact that a least favorable a priori weighting over ω_0 relative to a significance level α_0 may indeed depend on α_0 may be illustrated by the following simple example.

Example 4.1

S	$P(x_i \mid \omega_{01})$	$P(x_i \mid \omega_{02})$	$P(x_i \mid \omega_1)$
x_1	.05	.10	.6
x_2	.15	.05	.2
x_3	.8	.85	.2

In the above example the least favorable a priori weightings over $\omega_0 = \{\omega_{01}, \omega_{02}\}$ relative to significance levels $\alpha_0 = .1$ and $.2$ are $\eta^{(0)}(.1) = (0,1)$ and $\eta^{(0)}(.2) = (1,0)$ respectively. These assertions are easily verified by invoking the corollary to Theorem 3.

Theorem 3 does not assert that every most powerful α_0-test of $\omega_0(\eta^{(0)})$ against ω_1 (where $\eta^{(0)}$ is a least favorable weighting over ω_0 relative to a significance level α_0) is a solution to Problem 4.1. Example 4.2 illustrates this point.

Example 4.2

S	$P(x_i \mid \omega_{01})$	$P(x_i \mid \omega_{02})$	$P(x_i \mid \omega_1)$
x_1	28/48	36/48	1/3
x_2	14/48	2/48	1/3
x_3	6/48	10/48	1/3

For the above example, the least favorable weighting over ω_0 relative to a significance level of .01 is (1/2,1/2) which yields an auxiliary problem:

S	$P(x_i \mid \omega_0(1/2,1/2))$	$P(x_i \mid \omega_1)$
x_1	2/3	1/3
x_2	1/6	1/3
x_3	1/6	1/3

For this auxiliary problem, $\phi^{(0)} = (0,.03,\ .03)$ is a most powerful .01-test but $\phi^{(0)} \notin F$; however, $\phi^{(1)} = (0,\ .015,\ .045)$ is a most powerful .01-test and $\phi^{(1)} \in F$. Consequently, $\phi^{(1)}$ is a solution of the problem and (1/2,1/2) is the least favorable a priori weighting over ω_0 relative to the significance level .01.

The geometric interpretation of Problem 4.2' is similar to that of Problem 3.2 which was discussed in Section 3. To this end, consider the r+1 points in $E^{(m)}$ space, viz: $\underline{f}^{(j)} = (f_{1j},\ldots,f_{nj})$, $j = 1,2,\ldots,r$, and $\underline{g} = (g_1,g_2,\ldots,g_m)$. Let $V_j \underline{f}^{(j)}$ represent the convex hull of the points $\underline{f}^{(j)}$, $j = 1,2,\ldots,r$ and let $P+\underline{g}$ represent the closed positive orthant displaced by the vector $\underline{g}$. To each a priori weighting ξ we can associate the point $f(\xi) = (f_1(\xi),\ldots,f_n(\xi)) \in V_j \underline{f}^{(j)}$, and conversely each point of $V_j \underline{f}^{(j)}$ is associated with at least one a priori weighting ξ. Problem 4.2' is equivalent to the problem of finding the minimum distance from the origin to the displaced orthant $P+\underline{g}$ where the distance is taken under the constraints that one can only move parallel to the coordinate axes or along a single ray from the origin through some point of $V_j \underline{f}^{(j)}$; furthermore, the unit of measurement along this ray is such that the distance from the origin to the convex set $V_j \underline{f}^{(j)}$ is α_0. Stated slightly differently we are constrained to move parallel to the coordinate axes or along a single ray of the convex polyhedral cone which is the sum cone of the vectors $\underline{f}^{(j)}$, $j = 1,2,\ldots,r$. The ray of the cone associated with the path yielding the minimum distance is associated with some element of $V_j \underline{f}^{(j)}$ and thus with some $\xi^{(0)}$. This element $\xi^{(0)}$ is a least favorable a priori weighting over ω_0 relative to the significance level α_0.

5. Composite hypothesis versus composite alternative.

In this section we assume that $\omega_0 = \{\omega_{01}, \ldots, \omega_{0r}\}$ and $\omega_1 = \{\omega_{11}, \omega_{12}, \ldots, \omega_{1s}\}$. Let $f_{ij} = P\{x_i | \omega_{0j}\}$ and $g_{ik} = \{x_i | \omega_{1k}\}$ where $i = 1,2,\ldots,n$; $j = 1,2,\ldots,r$; $k = 1,2,\ldots,s$. The randomized strategy $\underline{\phi} = (\phi_1, \ldots, \phi_n)$ accepts ω_0 and ω_1 with probabilities $1 - \phi_i$ and ϕ_i, respectively, if x_i is observed.

Problem 5.1: Let F be the set of all $\underline{\phi}$ such that:

$$\text{i) } \phi_i \geqslant 0, \quad i = 1,2,\ldots,n,$$

$$\text{(5.1)} \quad \text{ii) } \phi_i \leqslant 1, \quad i = 1,2,\ldots,n,$$

$$\text{iii) } \alpha_j(\underline{\phi}) = \Sigma_i f_{ij} \phi_i \leqslant \alpha_0 \quad \text{for } j = 1,2,\ldots,r;$$

to find $\underline{\phi}^{(0)} \in F$ which is a maximizer of

$$\text{(5.2)} \qquad \min_k \left[1 - \beta_k(\underline{\phi})\right] = \min_k \left[\Sigma_i g_{ik} \phi_i\right]$$

(i.e., to find $\underline{\phi}^{(0)} \in F$ which maximizes the minimum power over the elements of ω_1).

Note that if instead of 5.2 we wished to maximize, say:

$$\text{(5.3)} \qquad \Sigma_k \left[\zeta_k \; \Sigma_i g_{ik} \phi_i\right]$$

where $\zeta_k \geqslant 0$, $\Sigma_k \zeta_k = 1$ then the altered Problem 5.2 would boil down to Problem 4.1 by simply collapsing the composite

alternative to a simple alternative according to the weights $(\xi_1, \ldots, \xi_k)$.

We alter Problem 5.1 slightly to yield:

Problem 5.1': Let F' be the set of all $\underline{\phi}$ such that $\underline{\phi}$ satisfies conditions (5.1) and in addition

$$(5.4) \qquad 1 - \beta_{\underline{\phi}}(k) = \sum_i g_{ik}\,\phi_i \leqslant M \quad \text{for } k = 1,2,\ldots,s;$$

to find the maximum M for which $F' \neq 0$ (empty set).

Expression (5.4) is equivalent to

$$(5.5) \qquad M - \sum_i g_{ik}\,\phi_i \leqslant 0 \qquad \text{for } k = 1,2,\ldots,s.$$

Finally we adjust Problem 5.1' to yield:

Problem 5.1'': Let F'' be the set of all pairs $(M, \underline{\phi})$ such that M is non-negative, $\underline{\phi}$ satisfies conditions (5.1) and $(M, \underline{\phi})$ satisfies (5.5) ; to find $(M^{(0)}, \underline{\phi}^{(0)}) \in F''$ which is a maximizer of M.

The dual of Problem 5.1'' can then be put in the form:

Problem 5.2: Let H be the set of elements $(\xi_1, \xi_2, \ldots, \xi_n; \rho, \underline{\eta}; \mu, \underline{\zeta})$ where

(5.6) i):

a) $\xi_i \geqslant 0$, $i = 1,2,\ldots,n$,

b) $\rho \geqslant 0$ and $\underline{\eta} = (\eta_1, \ldots, \eta_r)$ where $\sum_j \eta_j = 1$, $\eta_j \geqslant 0$ for $j = 1,2,\ldots,r$,

c) $\mu \geqslant 0$ and $\underline{\zeta} = (\zeta_1, \ldots, \zeta_s)$ where $\sum_k \zeta_k = 1$, $\zeta_k \geqslant 0$ for $k = 1,2,\ldots,s$,

ii) $\mu \geq 1$,

iii) $\xi_i + \rho f_i(\eta) \geq \mu \; g_i(\zeta)$ where

$$f_i(\eta) \equiv \Sigma_j f_{ij} \eta_j, \; g_i(\zeta) = \Sigma_k g_{ik} \zeta_k$$

for $i = 1, 2, \ldots, n$;

to find $(\xi_1^{(0)}, \ldots \xi_n^{(0)}; \rho^{(0)}, \eta^{(0)}; \mu^{(0)}, \zeta^{(0)}) \in H$ which is a minimizer of

$$\Sigma_i \xi_i + \alpha_0 \rho \tag{5.7}$$

It is clear that $\mu^{(0)} = 1$. For a given (η, ζ) we investigate how we should minimize (5.7) subject to the constraints (5.6). Since this restricted problem (for the specific (η, ζ) pair) can be identified with Problem 3.2, we are led to the following statistical problem corresponding to the dual of this restricted problem: Test $\omega_0(\eta)$ against $\omega_1(\zeta)$ where $P\left\{ x_i | \omega_0(\eta) \right\} = f_i(\eta)$ and $P\left\{ x_i | \omega_1(\zeta) \right\} = g_i(\zeta)$ for $i = 1, 2, \ldots, n$. Let $\phi(\eta, \zeta)$ be a most powerful α_0-test for this auxiliary problem, and let the power for this test be $\beta\left[\phi(\eta, \zeta)\right]$. From Section 3 we conclude that the solution of Problem 5.2 (for a fixed η, ζ) yields a minimum value for (5.7) equal to $1 - \beta\left[\phi(\eta, \zeta)\right]$. Hence to solve Problem 5.2 we must find $(\eta^{(0)}, \zeta^{(0)})$ which is a maximizer of $\beta\left[\phi(\eta, \zeta)\right]$. Note that existence questions follow readily since (η, ζ) can only take on values in a compact set. The pair $(\eta^{(0)}, \zeta^{(0)})$

is said to be a <u>least favorable a priori weighting over</u> (ω_0, ω_1) <u>relative to the significance level</u> α_0.

Let $\phi^{(0)}$ be a solution of Problem 5.1. Then from Theorem 1 c) we have

$$\text{(5.8)} \qquad \min_k \left[\sum_i \phi_i^{(0)} g_{ik}\right] = 1 - \beta \left[\phi(\eta^{(0)}, \xi^{(0)})\right].$$

Since $\phi^{(0)} \in F$, $\alpha_j(\phi^{(0)}) = \sum_i f_{ij}\phi_i^{(0)} \leq \alpha_0$ for $j = 1,2,\ldots,r$, we can conclude from (5.8) that $\phi^{(0)}$ is a most powerful α_0-test for $\omega_0(\eta^{(0)})$ against $\omega_1(\xi^{(0)})$. We summarize by means of the following theorem:

<u>Theorem 4</u>: If $\phi^{(0)}$ is a solution of Problem 5.1 then $\phi^{(0)}$ is a most powerful α_0-test for $\omega_0(\eta^{(0)})$ against $\omega_1(\xi^{(0)})$ where $(\eta^{(0)}, \xi^{(0)})$ is a least favorable a priori weighting over (ω_0, ω_1) relative to significance level α_0.

<u>Corollary</u>: For a given a priori weighting, say $(\eta^{(1)}, \xi^{(1)})$ over (ω_0, ω_1) let $\phi(\eta^{(1)}, \xi^{(1)})$ be a most powerful α_0-test for $\omega_0(\eta^{(1)})$ against $\omega_1(\xi^{(1)})$ yielding a β-value of $\beta[\phi(\eta^{(1)}, \xi^{(1)})] = \beta^{(1)}$, say. If $[1-\beta^{(1)}, \phi(\eta^{(1)}, \xi^{(1)})] \in F''$ then a) $\phi(\eta^{(1)}, \xi^{(1)})$ is a solution of Problem 5.1; b) $(\eta^{(1)}, \xi^{(1)})$ is a least favorable a priori weighting over (ω_0, ω_1) relative to the significance level α_0; and c) $\beta^{(1)} = \min_\phi \max_k \beta_k(\phi)$ where $\min_\phi$ is taken over all of F.

In particular, $[1-\beta^{(1)}, \not{g}(\eta^{(1)}, \xi^{(1)})] \in F''$ if and only if:

i) $\alpha_j[\not{g}(\eta^{(1)}, \xi^{(1)})] \leq \alpha_0$ for $j = 1,2,\ldots,r$ (condition 5.1, iii)

and

ii) $\beta_k[\not{g}(\eta^{(1)}, \xi^{(1)})] \leq \beta^{(1)}$ for $k = 1,2,\ldots,s$ (condition 5.5)

Proof of Corollary: The proof is an immediate consequence of of Theorem 1 (part d) as applied to Problems 5.1'' and 5.2.

The geometric interpretation of Problem 5.2 is as follows: Let $\underline{g}^{(k)} = (g_{1k},\ldots,g_{nk})$ for $k = 1,2,\ldots,s$ and let the convex hull of these points be donoted by $V_k\underline{g}^{(k)}$. Problem 5.2 is equivalent to finding the minimum distance from the origin to some displaced orthant $P + \underline{g}$, where $\underline{g} \in V_k\underline{g}^{(k)}$, along a path which is constrained to follow parallel to the coordinate axes and along a single ray of the polyhedral sum cone generated by $\underline{f}^{(j)}$, $j = 1,2,\ldots,r$; the unit of measurement along the ray of the cone is such that the distance from the origin to $V_j\underline{f}^{(j)}$ is α_0. A minimum path selects a ray of the cone which is associated to some $\eta^{(0)}$, and, the particular displaced orthant, say, $P + \underline{g}^{(0)}$ associated with the path yields some element $\underline{\xi}^{(0)}$ where $\underline{g}(\underline{\xi}^{(0)}) = \underline{g}^{(0)}$. The pair $(\eta^{(0)}, \underline{\xi}^{(0)})$ is then a least favorable a priori weighting over (ω_0, ω_1) relative to the significance level α_0.

5. Concluding Remarks:

Since problems of type 4.1 or 5.1'' are linear programming problems there are existing computational algorithms for obtaining solutions. One could consider Problem 5.1 in the form of Problem 5.1'' and use the simplex technique of Dantzig [2] and others; or, one could apply the techniques used in the paper by Motzkin, Raiffa, Thompson and Thrall [8] to handle the problem directly in the form of Problem 5.1 by first using the double description method to characterize F and then to build up the minimum polyhedral surface corresponding to expression (5.2).

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